STRING EQUATIONS OF THE Q-KP HIERARCHY

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Abstract. Based on the Lax operator L and Orlov-Shulman's M operator, the string equations of the q-KP hierarchy are established from special additional symmetry flows, and the negative Virasoro constraint generators $\{L_{-n}, n \geq 1\}$ of the $2-$reduced q-KP hierarchy are also obtained.

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1. Introduction

The q-deformed integrable system (also called q-analogue or q-deformation of classical integrable system) is defined by means of q-derivative ∂_q [1, 2] instead of usual derivative ∂ with respect to x in a classical system. It reduces to a classical integrable system as $q \to 1$. Recently, the q-deformed Kadomtsev-Petviashvili (q-KP) hierarchy is a subject of intensive study in the literature from [3] to [14]. Its infinite conservation laws, bi-Hamiltonian structure, τ function, additional symmetries and its constrained sub-hierarchy have already been reported in [4, 5, 11, 12, 14].

The additional symmetries, string equations and Virasoro constraints of the KP hierarchy are important as they are involved in the matrix models of the string theory [15]. For example, there are several new works [16–20] on this topic. The additional symmetries were discovered independently at least twice by Sato School [21] and Orlov-Shulman [22], in quite different environments and philosophy although they are equivalent essentially. It is well-known that L.A.Dickey [23] presented a very elegant and compact proof of Adler-Shiota-van Moerbeke (ASvM) formula [24,25] based on the Lax operator L and Orlov and Shulman's M operator [22], and gave the string equation and the action of the additional symmetries on the τ function of the classical KP hierarchy. S.Panda and S.Roy gave the Virasoro and W-constraints on the τ function of the p-reduced KP hierarchy by expanding the additional symmetry operator in terms of the Lax operator [26, 27]. It is quite interesting to study the analogous properties of q-deformed KP hierarchy by this expanding method . The main purpose of this article is to give the string equations of the q-KP hierarchy, and then study the negative Virasoro constraint generators $\{L_{-n}, n \geq 1\}$ of 2-reduced q-KP hierarchy.

The organization of this paper is as follows. We recall some basic results and additional symmetries of q-KP hierarchy in Section 2. The string equations are given in Sections 3. The Virasoro constraints on the τ function of the 2-reduced (q-KdV) hierarchy are studied in Section 4. Section 5 is devoted to conclusions and discussions.

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At the end of the this section, we shall collect some useful facts of q-calculus [2] to make this paper be self-contained. The q-derivative ∂_q is defined by

$$\partial_q(f(x)) = \frac{f(qx)-f(x)}{x(q-1)} \tag{1.1}$$

and the q-shift operator is

$$\theta(f(x)) = f(qx). \tag{1.2}$$

$\partial_q(f(x))$ recovers the ordinary differentiation $\partial_x(f(x))$ as q goes to 1. Let ∂_q^{-1} denote the formal inverse of ∂_q. In general the following q-deformed Leibnitz rule holds

$$\partial_q^n \circ f = \sum_{k\geq 0} \binom{n}{k}_q \theta^{n-k}(\partial_q^k f)\partial_q^{n-k}, \qquad n\in\mathbb{Z} \tag{1.3}$$

where the q-number and the q-binomial are defined by

$$(n)_q = \frac{q^n-1}{q-1},$$

$$\binom{n}{k}_q = \frac{(n)_q(n-1)_q\cdots(n-k+1)_q}{(1)_q(2)_q\cdots(k)_q}, \quad \binom{n}{0}_q = 1.$$

For a q-pseudo-differential operator(q-PDO) of the form $P=\sum_{i=-\infty}^n p_i\partial_q^i$, we separate P into the differential part $P_+ = \sum_{i\geq 0} p_i\partial_q^i$ and the integral part $P_- = \sum_{i\leq -1} p_i\partial_q^i$. The conjugate operation "$*$" for P is defined by $P^* = \sum_i (\partial_q^*)^i p_i$ with $\partial_q^* = -\partial_q\theta^{-1} = -\frac{1}{q}\partial_{\frac{1}{q}}$, $(\partial_q^{-1})^* = (\partial_q^*)^{-1} = -\theta\partial_q^{-1}$.

The q-exponent e_q^x is defined as follows

$$e_q^x = \sum_{n=0}^\infty \frac{x^n}{(n)_q!}, \quad (n)_q! = (n)_q(n-1)_q(n-2)_q\cdots(1)_q.$$

Its equivalent expression is of the form

$$e_q^x = \exp(\sum_{k=1}^\infty \frac{(1-q)^k}{k(1-q^k)}x^k), \tag{1.4}$$

which is crucial to develop the τ function of the q-KP hierarchy [11].

2. q-KP hierarchy and its additional symmetries

Similar to the general way of describing the classical KP hierarchy [21, 28], we first give a brief introduction of q-KP hierarchy and its additional symmetries based on [11, 12].

Let L be one q-PDO given by

$$L = \partial_q + u_0 + u_{-1}\partial_q^{-1} + u_{-2}\partial_q^{-2} + \cdots, \tag{2.1}$$

which are called Lax operator of q-KP hierarchy. There exist infinite number of q-partial differential equations related to dynamical variables $\{u_i(x,t_1,t_2,t_3,\cdots,), i=0,-1,-2,-3,\cdots\}$ and can be deduced from the generalized Lax equation,

$$\frac{\partial L}{\partial t_n} = [B_n, L], \quad n=1,2,3,\cdots, \tag{2.2}$$

which are called q-KP hierarchy. Here $B_n = (L^n)_+ = \sum\limits_{i=0}^{n} b_i \partial_q^i$ and $L_-^n = L^n - L_+^n$. L in eq. (2.1) can be generated by dressing operator $S = 1 + \sum_{k=1}^{\infty} s_k \partial_q^{-k}$ in the following way

$$L = S \circ \partial_q \circ S^{-1}. \tag{2.3}$$

Dressing operator S satisfies Sato equation

$$\frac{\partial S}{\partial t_n} = -(L^n)_- S, \quad n = 1, 2, 3, \cdots. \tag{2.4}$$

The q-wave function $w_q(x,t;z)$ and the q-adjoint function $w_q^*(x,t;z)$ are given by

$$w_q = S e_q^{xz} \exp(\sum_{i=1}^{\infty} t_i z^i),$$

$$w_q^*(x,t;z) = (S^*)^{-1}|_{x/q} e_{1/q}^{-xz} \exp(-\sum_{i=1}^{\infty} t_i z^i),$$

which satisfies following linear q-differential equations

$$L w_q = z w_q, \quad L^*|_{x/q} w_q^* = z w_q^*.$$

Here the notation $P|_{x/t} = \sum_i P_i(x/t) t^i \partial_q^i$ is used for $P = \sum_i p_i(x) \partial_q^i$.

Furthermore, $w_q(x,t;z)$ and $w_q^*(x,t;z)$ can be expressed by sole function $\tau_q(x;t)$ [11] as

$$w_q = \frac{\tau_q(x; t - [z^{-1}])}{\tau_q(x;t)} e_q^{xz} \exp\left(\sum_{i=1}^{\infty} t_i z^i\right) = \frac{e_q^{xz} e^{\xi(t,z)} e^{-\sum_{i=1}^{\infty} \frac{z^{-i}}{i} \partial_i} \tau_q}{\tau_q}, \tag{2.5}$$

$$w_q^* = \frac{\tau_q(x; t + [z^{-1}])}{\tau_q(x;t)} e_{1/q}^{-xz} \exp\left(-\sum_{i=1}^{\infty} t_i z^i\right) = \frac{e_{1/q}^{-xz} e^{-\xi(t,z)} e^{+\sum_{i=1}^{\infty} \frac{z^{-i}}{i} \partial_i} \tau_q}{\tau_q},$$

where

$$[z] = \left(z, \frac{z^2}{2}, \frac{z^3}{3}, \dots\right).$$

The following Lemma shows there exist an essential correspondence between q-KP hierarchy and KP hierarchy.

Lemma 1. [11] Let $L_1 = \partial + u_{-1}\partial^{-1} + u_{-2}\partial^{-2} + \cdots$, where $\partial = \partial/\partial x$, be a solution of the classical KP hierarchy and τ be its tau function. Then

$$\tau_q(x,t) = \tau(t + [x]_q)$$

is a tau function of the q-KP hierarchy associated with Lax operator L in eq. (2.1), where

$$[x]_q = \big(x, \frac{(1-q)^2}{2(1-q^2)} x^2, \frac{(1-q)^3}{3(1-q^3)} x^3, \cdots, \frac{(1-q)^i}{i(1-q^i)} x^i, \cdots\big).$$

Define Γ_q and Orlov-Shulman's M operator

$$\Gamma_q = \sum_{i=1}^{\infty} \Big(i t_i + \frac{(1-q)^i}{(1-q^i)} x^i\Big) \partial_q^{i-1}, \tag{2.6}$$

$$M = S\Gamma_q S^{-1}. \tag{2.7}$$

Dressing $[\partial_k - \partial_q^k, \Gamma_q]$=0 gives

$$\partial_k M = [B_k, M]. \tag{2.8}$$

Eq. (2.2) together with eq. (2.8) implies that

$$\partial_k (M^m L^n) = [B_k, M^m L^n]. \tag{2.9}$$

Define the additional flows for each pair m, n as follows

$$\frac{\partial S}{\partial t^*_{m,n}} = -(M^m L^n)_- S, \tag{2.10}$$

or equivalently

$$\frac{\partial L}{\partial t^*_{m,n}} = -[(M^m L^n)_-, L], \tag{2.11}$$

$$\frac{\partial M}{\partial t^*_{m,n}} = -[(M^m L^n)_-, M]. \tag{2.12}$$

The additional flows $\partial^*_{mn} = \dfrac{\partial}{\partial t^*_{m,n}}$ commute with the hierarchy, i.e. $[\partial^*_{mn}, \partial_k] = 0$ but do not commute with each other, so they are additional symmetries [12]. $(M^m L^n)_-$ serves as the generator of the additional symmetries along the trajectory parametrized by $t^*_{m,n}$.

3. String equations of the q-KP hierarchy

In this section we shall get string equations for the q-KP hierarchy from special additional symmetry flows. For this, we need a lemma.

Lemma 2. The following equation

$$[M, L] = -1 \tag{3.1}$$

holds.

Proof. Direct calculations show that

$$\begin{aligned}
[\Gamma_q, \partial_q] &= \left[\sum_{i=1}^{\infty} \left(it_i + \frac{(1-q)^i}{1-q^i} x^i\right) \partial_q^{i-1}, \partial_q\right] \\
&= \sum_{i=1}^{\infty} \left[\frac{(1-q)^i}{1-q^i} x^i \partial_q^{i-1}, \partial_q\right] \\
&= \sum_{i=1}^{\infty} \frac{(1-q)^i}{1-q^i} \left(x^i \partial_q^i - (\partial_q \circ x^i) \partial_q^{i-1}\right) \\
&= \sum_{i=1}^{\infty} \frac{(1-q)^i}{1-q^i} \left(x^i \partial_q^i - ((\partial_q x^i) + q^i x^i \partial_q) \partial_q^{i-1}\right) \\
&= \sum_{i=1}^{\infty} \frac{(1-q)^i}{1-q^i} \left((1-q^i) x^i \partial_q^i - \frac{1-q^i}{1-q} x^{i-1} \partial_q^{i-1}\right)
\end{aligned}$$

$$= \sum_{i=1}^{\infty}((1-q)^i x^i \partial_q^i - (1-q)^{i-1} x^{i-1} \partial_q^{i-1})$$
$$= -1,$$

where we have used $[t_i, \partial_q] = 0$ in the second step and $\partial_q \circ x^i = (\partial_q x^i) + q^i x^i \partial_q$ in the fourth step. Then

$$[M, L] = [S\Gamma_q S^{-1}, S\partial_q S^{-1}] = S[\Gamma_q, \partial_q]S^{-1} = -1.$$

□

By virtue of Lemma 2, we have

Corollary 1. $[M, L] = -1$ implies $[M, L^n] = -nL^{n-1}$. Therefore,

$$[ML^{-n+1}, L^n] = -n. \tag{3.2}$$

The action of additional flows $\partial^*_{1,-n+1}$ on L^n are $\partial^*_{1,-n+1} L^n = -[(ML^{-n+1})_-, L^n]$, which can be written as

$$\partial^*_{1,-n+1} L^n = [(ML^{-n+1})_+, L^n] + n. \tag{3.3}$$

The following theorem holds by virtue of eq.(3.3).

Theorem 1. If an operator L does not depend on the parameters t_n and the additional variables $t^*_{1,-n+1}$, then L^n is a purely differential operator, and the string equations of the q-KP hierarchy are given by

$$[L^n, \frac{1}{n}(ML^{-n+1})_+] = 1, \; n = 2, 3, 4, \cdots \tag{3.4}$$

In view of the additional symmetries and string equations, we can get the following corollary, which plays a crucial role in the study of the constraints on the τ function of the p-reduced q-KP hierarchy.

Corollary 2. If L^n is a differential operator, and $\partial^*_{1,-n+1} S = 0$, then

$$(ML^{-n+1})_- = \frac{n-1}{2} L^{-n}, \; n = 2, 3, 4, \cdots \tag{3.5}$$

Proof. Since $[M, L] = -1$, it is not difficult to obtain

$$[M, L^{-n+1}] = (n-1)L^{-n},$$

and hence

$$(ML^{-n+1})_- - (L^{-n+1}M)_- = (n-1)L^{-n}. \tag{3.6}$$

Noticing $[(n-1)L^{-n}, L^n] = 0$, then

$$[(ML^{-n+1})_- - (L^{-n+1}M)_-, L^n] = 0, \quad \text{i.e.,}$$
$$[(ML^{-n+1})_-, L^n] = [(L^{-n+1}M)_-, L^n].$$

Thus

$$\partial^*_{1,-n+1} L^n = -[(L^{-n+1}M)_-, L^n]$$
$$= -\tfrac{1}{2}[(ML^{-n+1})_- + (L^{-n+1}M)_-, L^n],$$

or equivalently

$$\partial^*_{1,-n+1}S = -\frac{1}{2}(ML^{-n+1} + L^{-n+1}M)_- S.$$

Therefore, it follows from the equation $\partial^*_{1,-n+1}S = 0$ that

$$(ML^{-n+1} + L^{-n+1}M)_- = 0.$$

Combining this with (3.6) finishes the proof. □

4. Constraints on the τ function of the q-KdV hierarchy

In this section, we mainly study the associated constraints on τ-function of the 2-reduced q-KP (q-KdV) hierarchy from string equations eq. (3.4). To this end, we first define residue $\operatorname{res} L = u_{-1}$ of L given by eq. (2.1) and state two very useful lemmas.

Lemma 3. For $n = 1, 2, 3, \cdots$,

$$\operatorname{res} L^n = \frac{\partial^2 \log \tau_q}{\partial t_1 \partial t_n}. \tag{4.1}$$

where τ_q is the *tau* function of the q-KP hierarchy.

Proof. Taking the residue of $\frac{\partial S}{\partial t_n} = -(L^n)_- S$, we get

$$\frac{\partial s_1}{\partial t_n} = -\operatorname{res}((L^n)_-(1 + s_1\partial_q^{-1} + s_2\partial_q^{-2} + \cdots)) = -\operatorname{res}(L^n)_- = -\operatorname{res} L^n.$$

Noting that $u_0 = s_1 - \theta(s_1) = -x(q-1)\partial_q s_1 = x(q-1)\partial_q\partial_{t_1}\log\tau_q,\ \ s_1 = -\frac{\partial \log \tau_q}{\partial t_1}$ (see [14]), then

$$\operatorname{res} L^n = -\frac{\partial s_1}{\partial t_n} = \frac{\partial^2 \log \tau_q}{\partial t_1 \partial t_n}.$$

□

Lemma 4. Orlov-Shulman's M operator has the expansion of the form

$$M = \sum_{i=1}^{\infty}\left(it_i + \frac{(1-q)^i}{(1-q^i)}x^i\right)L^{i-1} + \sum_{i=1}^{\infty} V_{i+1}L^{-i-1}, \tag{4.2}$$

where

$$V_{i+1} = -i\sum_{a_1+2a_2+3a_3+\cdots=i}(-1)^{a_1+a_2+\cdots}\frac{(\partial t_1)^{a_1}}{a_1!}\frac{(\frac{1}{2}\partial t_2)^{a_2}}{a_2!}\frac{(\frac{1}{3}\partial t_3)^{a_3}}{a_3!}\cdots\log\tau_q.$$

Proof. First, we assert $Mw_q = \frac{\partial w_q}{\partial z}$. Indeed, from the identity $\partial_q^{i-1}e_q^{xz} = z^{i-1}e_q^{xz}$ we have that

$$Mw_q = S\Gamma_q S^{-1}Se_q^{xz}e^{\xi(t,z)} = S\left(\sum_{i=1}^{\infty}\left(it_i + \frac{(1-q)^i}{1-q^i}x^i\right)z^{i-1}\right)e_q^{xz}e^{\xi(t,z)},$$

where $\xi(t,z) = \sum_{i=1}^{\infty} t_i z^i$. On the other hand,

$$\begin{aligned}\frac{\partial w_q}{\partial z} &= \frac{\partial(Se_q^{xz}e^{\xi(t,z)})}{\partial z} = S\Big(\frac{\partial e_q^{xz}}{\partial z}e^{\xi(t,z)} + e_q^{xz}\frac{\partial e^{\xi(t,z)}}{\partial z}\Big)\\ &= S\left(\sum_{i=1}^{\infty}\left(it_i + \frac{(1-q)^i}{1-q^i}x^i\right)z^{i-1}\right)e_q^{xz}e^{\xi(t,z)}.\end{aligned}$$

Thus the assertion is verified. Next, by a direct calculation from eq.(1.4) and eq.(2.5), we have

$$\log w_q = \sum_{k=1}^{\infty} \frac{(1-q)^k}{k(1-q^k)}(xz)^k + \sum_{n=1}^{\infty} t_n z^n + \sum_{N=0}^{\infty} \frac{1}{N!}(-\sum_{i=1}^{\infty} \frac{z^{-i}}{i}\partial_i)^N \log \tau_q - \log \tau_q. \tag{4.3}$$

Let $M = \sum_{n=1}^{\infty} a_n L^{n-1} + \sum_{n=1}^{\infty} b_n L^{-n}$. Then in light of $Lw_q = zw_q$ and the assertion mentioned in above, we obtain

$$\frac{\partial w_q}{\partial z} = Mw_q = (\sum_{n=1}^{\infty} a_n L^{n-1} + \sum_{n=1}^{\infty} b_n L^{-n})w_q,$$

and hence

$$\frac{\partial \log w_q}{\partial z} = \frac{1}{w_q}\frac{\partial w_q}{\partial z} = \sum_{n=1}^{\infty} a_n z^{n-1} + \sum_{n=1}^{\infty} b_n z^{-n}. \tag{4.4}$$

Thus by comparing the coefficients of z in $\frac{\partial \log w_q}{\partial z}$ given by eq. (4.3) and eq. (4.4), a_i and b_i are determined such that M is obtained as eq. (4.2). □

To be an intuitive glance, the first few V_{i+1} are given as follows.

$$V_2 = \frac{\partial \log \tau_q}{\partial t_1},$$

$$V_3 = \frac{\partial \log \tau_q}{\partial t_2} - \frac{\partial^2 \log \tau_q}{\partial t_1^2},$$

$$V_4 = (\frac{1}{2}\frac{\partial^3}{\partial t_1^3} - \frac{3}{2}\frac{\partial^2}{\partial t_1 \partial t_2} + \frac{\partial}{\partial t_3}) \log \tau_q,$$

$$V_5 = (-\frac{1}{3!}\frac{\partial^4}{\partial t_1^4} - \frac{1}{2}\frac{\partial^2}{\partial t_2^2} - \frac{4}{3}\frac{\partial^2}{\partial t_1 \partial t_3} + \frac{\partial}{\partial t_4}) \log \tau_q,$$

$$V_6 = (\frac{1}{4!}\frac{\partial^5}{\partial t_1^5} - \frac{5}{12}\frac{\partial^4}{\partial t_1^3 \partial t_3} + \frac{5}{6}\frac{\partial^3}{\partial t_1^2 \partial t_3} - \frac{5}{4}\frac{\partial^2}{\partial t_1 \partial t_4} - \frac{5}{6}\frac{\partial^2}{\partial t_2 \partial t_3} + \frac{\partial}{\partial t_5}) \log \tau_q.$$

Now we consider the 2-reduced q-KP hierarchy(q-KdV hierarchy), by setting $L_-^2 = 0$ or setting

$$L^2 = \partial_q^2 + (q-1)xu\partial_q + u. \tag{4.5}$$

To make the following theorem be a compact form, introduce

$$L_{-n} = \frac{1}{2}\sum_{\substack{i=2n+1 \\ i \neq 0 (\mathrm{mod}\, 2)}}^{\infty} i\tilde{t}_i \frac{\partial}{\partial \tilde{t}_{i-2n}} + \frac{1}{4}\sum_{k+l=n+1}(2k-1)(2l-1)\tilde{t}_{2k-1}\tilde{t}_{2k-1} \tag{4.6}$$

and

$$\tilde{t}_i = t_i + \frac{(1-q)^i}{i(1-q^i)}x^i, \ \ i = 1, 2, 3, \cdots . \tag{4.7}$$

Theorem 2. If L^2 satisfies eq. (3.4), the Virasoro constraints imposed on the τ-function of the q-KdV hierarchy are

$$L_{-n}\tau_q = 0, \quad n = 1, 2, 3, \cdots , \tag{4.8}$$

and the Virasoro commutation relations

$$[L_{-n}, L_{-m}] = (-n+m)L_{-(n+m)}, \ \ m, n = 1, 2, 3, \cdots \tag{4.9}$$

hold.

Proof. For $n=1,2,3,\cdots$, we have

$$\operatorname{res}(ML^{-2n+1})=\operatorname{res}(ML^{-2n+1})_-=\operatorname{res}(-\frac{2n+1}{2}L^{-2n})_-=0 \tag{4.10}$$

with the help of eq. (3.5). Substituting the expansion of M in eq. (4.2) into eq. (4.10), then

$$\sum_{i=1}^{\infty}\Big(it_i+\frac{(1-q)^i}{1-q^i}x^i\Big)\operatorname{res}L^{i-2n}+\sum_{i=1}^{\infty}\operatorname{res}(V_{i+1}L^{-i-2n})=0,$$

which implies

$$\sum_{\substack{i=2n+1\\ i\neq 0(\operatorname{mod}2)}}^{\infty}\Big(it_i+\frac{(1-q)^i}{1-q^i}x^i\Big)\operatorname{res}L^{i-2n}+(2n-1)t_{2n-1}+\frac{(1-q)^{2n-1}}{1-q^{2n-1}}x^{2n-1}=0. \tag{4.11}$$

Substituting $\operatorname{res}L^{i-2n}=\frac{\partial^2\log\tau_q}{\partial t_1\partial t_{i-2n}}$ into eq. (4.11), then performing an integration with respect to t_1 and multiplying by $\frac{\tau_q}{2}$, it becomes

$$\tilde{L}_{-n}\tau_q=0,\quad n=1,2,3,\cdots,$$

where

$$\begin{aligned}\tilde{L}_{-n}=\frac{1}{2}&\sum_{\substack{i=2n+1\\ i\neq 0(\operatorname{mod}2)}}^{\infty}\Big(it_i+\frac{(1-q)^i}{1-q^i}x^i\Big)\frac{\partial}{\partial t_{i-2n}}+\frac{(1-q)^{2n-1}}{1-q^{2n-1}}\cdot\frac{1}{2}t_1x^{2n-1}\\ &+\frac{1}{2}(2n-1)t_1t_{2n-1}+C(t_2,t_3,\cdots;x).\end{aligned} \tag{4.12}$$

The integration constant $C(t_2,t_3,\cdots;x)$ with respect to t_1 could be the arbitrary function with the parameters $(t_2,t_3,\cdots;x)$. What we will do is to determine $C(t_2,t_3,\cdots;x)$ such that $\tilde{L}_{-n}$ satisfy Virasoro commutation relations.

Let

$$\tilde{t}_i=t_i+\frac{(1-q)^i}{i(1-q^i)}x^i,\ i=1,2,3,\cdots,$$

and choose $C(t_2,t_3,\cdots;x)$ as

$$\begin{aligned}C(t_2,t_3,\cdots;x)=-\frac{1}{4}\sum_{k=3}^{2n-3}&(2k-1)(2n-2k+1)\Big(t_{2k-1}+\frac{(1-q)^{2k-1}}{(2k-1)(1-q^{2k-1})}x^{2k-1}\Big)\\ &\cdot\Big(t_{2n-2k+1}+\frac{(1-q)^{2n-2k+1}}{(2n-2k+1)(1-q^{2n-2k+1})}x^{2n-2k+1}\Big)\\ &-\frac{1}{2}(2n-1)x\Big(t_{2n-1}+\frac{(1-q)^{2n-1}}{(2n-1)(1-q^{2n-1})}x^{2n-1}\Big),\end{aligned}$$

Then

$$\tilde{L}_{-n}=\frac{1}{2}\sum_{\substack{i=2n+1\\ i\neq 0(\operatorname{mod}2)}}^{\infty}i\tilde{t}_i\frac{\partial}{\partial\tilde{t}_{i-2n}}+\frac{1}{4}\sum_{k+l=n+1}(2k-1)(2l-1)\tilde{t}_{2k-1}\tilde{t}_{2k-1}\equiv L_{-n}$$

and

$$L_{-n}\tau_q=0,\quad n=1,2,3,\cdots$$

as we expected. By a straightforward and tedious calculation, the Virasoro commutation relations

$$[L_{-n}, L_{-m}] = (-n+m)L_{-(n+m)}, m, n = 1, 2, 3, \cdots$$

can be verified. □

Remark 1. As we know, the q-deformed KP hierarchy reduces to the classical KP hierarchy when $q \to 1$ and $u_0 = 0$. The parameters $(\tilde{t}_1, \tilde{t}_2, \cdots, \tilde{t}_i, \cdots)$ in eq. (4.6) tend to $(t_1 + x, t_2, \cdots, t_i, \cdots)$ as $q \to 1$. One can further identify $t_1 + x$ with x in the classical KP hierarchy, i.e. $t_1 + x \to x$, therefore the Virasoro generators L_{-n} in eq. (4.6) of the 2-reduced q-KP hierarchy tend to

$$\hat{L}_{-n} = \frac{1}{2} \sum_{\substack{i=2n+1 \\ i \neq 0 (\mathrm{mod}\, 2)}}^{\infty} it_i \frac{\partial}{\partial t_{i-2n}} + \frac{1}{4} \sum_{k+l=n+1} (2k-1)(2l-1)t_{2k-1}t_{2k-1}, n = 2, 3, \cdots \tag{4.13}$$

and

$$\hat{L}_{-1} = \frac{1}{2} \sum_{\substack{i=3 \\ i \neq 0 (\mathrm{mod}\, 2)}}^{\infty} it_i \frac{\partial}{\partial t_{i-2}} + \frac{1}{4} x^2, \tag{4.14}$$

which are identical with the results of the classical KP hierarchy given by L.A.Dickey [29] and S.Panda, S.Roy [26].

5. Conclusions and discussions

To summarize, we have derived the string equations in eq. (3.4) and the negative Virasoro constraint generators on the τ function of 2-reduced q-KP hierarchy in eq. (4.8) in Theorem 2. The results of this paper show obviously that the Virasoro generators $\{L_{-n}, n \geq 1\}$ of the q-KP hierarchy are different with the $\{\hat{L}_{-n}, n \geq 1\}$ of the KP hierarchy, although they satisfy the common Virasoro commutation relations. Furthermore, one can find the following interesting relation between the q-KP hierarchy and the KP hierarchy

$$L_{-n} = \hat{L}_{-n}|_{t_i \to \tilde{t}_i = t_i + \frac{(1-q)^i}{i(1-q^i)} x^i},$$

and it seems to demonstrate that q-deformation is a non-uniform transformation for coordinates $t_i \to \tilde{t}_i$, which is consistent with results on τ function [11] and the q-soliton [14] of the q-KP hierarchy.

For the p-reduced ($p \geq 3$) q-KP hierarchy, which is the q-KP hierarchy satisfying the reduction condition $(L^p)_- = 0$, we can obtain $(ML^{pn+1})_- = 0$. Using the similar technique in q-KdV hierarchy, we can deduce the Virasoro constraints on the τ function of the p-reduced q-KP hierarchy for $p \geq 3$. Moreover, for $\{L_n, n \geq 0\}$ we find a subtle point at the calculation of $res(V_{i+1}L^{-i+2n})$, and will try to study it in the future.

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